

Uniform Quantification of Correlations for Bipartite Systems

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Based on the relative entropy, we give a unified characterization of quantum correlations for nonlocality, steerability, discord and entanglement for any bipartite quantum states. For two-qubit states we show that the quantities obtained from quantifying nonlocality, steerability, entanglement and discord have strictly monotonic relationship. As for examples, the Bell diagonal states are studied in detail.

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INTRODUCTION

As a key feature of quantum mechanics, quantum correlation has many different forms such as entanglement, discord, steering, nonlocality etc.. Nonlocality was first pointed out in 1935 by Einstein, Podolsky and Rosen (EPR) [1], indicating that the nonlocality must be an artefact of the incompleteness of quantum mechanics. An interesting response to the EPR paradox was given by Schrödinger [2], who introduced another kind of correlation in entangled states - quantum steering: Alice's ability to affect Bob's state through her choice of measure basis. Later, it was shown that the measurement outcomes from local measurements on entangled states may be nonlocal, in the sense that they violate a Bell inequality [3]. Let us briefly recall their definitions.

Quantum entanglement: Let ρ_{AB} be a bipartite state with subsystems A and B. ρ_{AB} is said to be separable if it can be expressed as

$$\rho_{AB} = \sum_{i=1}^n p_i \rho_i^A \otimes \rho_i^B,$$

where $0 \leq p_i \leq 1$, $\sum_i p_i = 1$, ρ_i^A and ρ_i^B are states of the subsystems A and B, respectively. Otherwise it is called entangled. Basic aspects of entanglement including the characterization, detection, distillation and quantification have been reviewed in [4].

Bell nonlocality: Let \mathcal{M}_a (\mathcal{M}_b) denote the set of observables that Alice (Bob) performs measurement on the systems A (B). Let $\lambda(A)$ ($\lambda(B)$) stand for the eigenvalues $\{a\}$ ($\{b\}$) of $A \in \mathcal{M}_a$ ($B \in \mathcal{M}_b$), and $P(a|A; W)$ ($P(b|B; W)$) the probability that Alice (Bob) gets the measurement outcome a (b) when she (he) measures the subsystem A (B) of the state W . We say that a state W is Bell local, if the following relation is satisfied for all $a \in \lambda(A)$, $b \in \lambda(B)$, $A \in \mathcal{M}_a$ and $B \in \mathcal{M}_b$,

$$\begin{aligned} & \text{Tr}[(\Pi_a^A \otimes \Pi_b^B)W] \\ & \equiv P(a, b|A, B; W) = \sum_{\xi} p(a|A, \xi) p(b|B, \xi) p_{\xi}, \end{aligned}$$

where Π_a^A (Π_b^B) is the projector satisfying $A \Pi_a^A = a \Pi_a^A$ ($B \Pi_b^B = b \Pi_b^B$), $p(a|A, \xi)$ and $p(b|B, \xi)$ are some probability distributions involving the local hidden variable (LHV) ξ . Recently, it has been realized that one can significantly expand the notion of quantum nonlocality by considering more complex causal structures going beyond the usual LHV models [5, 6].

Quantum steering: If Alice performs the measurement x and obtains the outcome a , then Bobs subnormalized reduced state is given by $\rho_{a|x} = \text{tr}_A[(A_{a|x} \otimes I)\rho]$. And $\sum_a \rho_{a|x} = \rho_B$ is independent of the measurement chosen by Alice. The whole collection of ensembles $\{\rho_{a|x}\}_{a,x}$ is a state assemblage. If there exists a local hidden state (LHS) model such that

$$\rho_{a|x} = \sum_{\lambda} p(\lambda) p(a|x, \lambda) \sigma_{\lambda},$$

then Alice can not steer Bob's system. Here, $p(a|x, \lambda)$ are some conditional probability distributions, and σ_{λ} are a collection of subnormalized states that sum up to ρ_B and satisfy $\sum_{\lambda} \text{tr}[p(\lambda) \sigma_{\lambda}] = 1$ [7–9]. The steering problem is closely related to the joint-measurement problem [10, 11].

Quantum discord: Two classically identical expressions for the mutual information generally differ when the systems involved are quantum. This difference was defined as the quantum discord. It can be used as a measure of the quantumness of correlations [12]. Later Ref. [13] gives the mathematical definition of quantum discord of a state ρ ,

$$\mathcal{D}(\rho) = \min_{\sigma \in CC} S(\rho \parallel \sigma),$$

where CC stands for the set of classically correlated states of the form $\sum_{ij} p_{ij} |i\rangle\langle i| \otimes |j\rangle\langle j|$, where p_{ij} is a joint probability distribution and $|i\rangle$ span the local orthonormal basis.

Although our understanding on different kinds of correlations has advanced greatly recently, many fundamental questions remain open, e.g., (1) How to quantify these correlations? (2) What is the relationship between them?

To the first question there are already a lot of literatures, see Ref.[4, 20] for entanglement of quantum states, Ref.[12, 13, 21] for quantifying quantum discord, Ref. [22–25] for the measure of nonlocality, and Ref. [26–28] for quantifying steering. There are many measures for different correlations. And the relationship between them is rather complicated. It is even difficult to compare the measures for a given correlation. Thus, one would ask if there is a unified quantification for all the above correlations.

In this work, we give a unified quantification for all these quantum correlations. We first give the definitions of relative entropy steering and relative entropy Nonlocality. Then we study the relationship among them for two-qubit states. At last, we discuss multipartite situations.

UNIFIED QUANTIFICATION OF QUANTUM CORRELATIONS

The von Neumann relative entropy is defined as [29] $S(\rho||\sigma) \equiv \text{tr}\{\rho(\ln \rho - \ln \sigma)\}$ (in our text, we take \log_2 instead of \ln). In fact, Vedral et al. [30, 31] first introduced the relative entropy of entanglement, while the relative entropy of discord was first proposed by Modi et al. [13]. Let us list the above definitions [19]:

$$\mathcal{E}(\rho) = \min_{\sigma \in \mathcal{S}} S(\rho || \sigma),$$

$$\mathcal{D}(\rho) = \min_{\sigma \in \mathcal{CC}} S(\rho || \sigma),$$

where $\mathcal{E}(\rho)$ and $\mathcal{D}(\rho)$ are quantum entanglement and quantum discord of state ρ , \mathcal{S} and \mathcal{CC} stand for the sets of separable states and classically correlated states, respectively.

In the following, we choose von Neumann relative entropy to measure the quantum steerability and nonlocality of quantum states. We first give a lemma.

Lemma 1. *The sets of all unsteerable states and LHV states are convex sets, respectively.*

Proof. Any state admitting LHV models does not violate any Bell inequality. Let ρ_1 and ρ_2 be two such LHV states. They satisfy all Bell inequalities like $\text{tr}(\mathcal{B}\rho_1) \leq c$ and $\text{tr}(\mathcal{B}\rho_2) \leq c$, where \mathcal{B} is any Bell operators and c some constant. Then one has $\text{tr}[\mathcal{B}(s\rho_1 + (1-s)\rho_2)] = s[\text{tr}(\mathcal{B}\rho_1)] + (1-s)[\text{tr}(\mathcal{B}\rho_2)] \leq c$, where $s \in [0, 1]$. This proves that the LHV states constitute a convex set.

Suppose that states ρ and $\tilde{\rho}$ are unsteerable. From (1), there exists LHS model such that

$$\rho_{a|x} = \text{tr}_A[(A_{a|x} \otimes I)\rho] = \sum_{\lambda} p(\lambda)p(a|x, \lambda)\sigma_{\lambda},$$

where $p(a|x, \lambda) \geq 0$, and σ_{λ} are a collection of subnormalized states that sum up to ρ_B which satisfies $\sum_{\lambda} \text{tr}[p(\lambda)\sigma_{\lambda}] = 1$. Analogously for $\tilde{\rho}$,

$$\tilde{\rho}_{a|x} = \text{tr}_A[(A_{a|x} \otimes I)\tilde{\rho}] = \sum_{\mu} q(\mu)q(a|x, \mu)\sigma_{\mu}.$$

Then

$$\begin{aligned} [s\rho + (1-s)\tilde{\rho}]_{a|x} &= \text{tr}_A[(A_{a|x} \otimes I)[s\rho + (1-s)\tilde{\rho}]] \\ &= s[\text{tr}_A[(A_{a|x} \otimes I)\rho]] + (1-s)\text{tr}_A[(A_{a|x} \otimes I)\tilde{\rho}] \\ &= s \sum_{\lambda} p(\lambda)p(a|x, \lambda)\sigma_{\lambda} + (1-s) \sum_{\mu} q(\mu)q(a|x, \mu)\sigma_{\mu} \\ &= \sum_{\nu=1}^{m+n} p(\nu)p(a|x, \nu)\sigma_{\nu}, \end{aligned}$$

where, $p(\nu) = sp(\lambda)$, $\sigma_{\nu} = \sigma_{\lambda}$ and $\nu = \lambda$ for $\nu = 1, 2, \dots, m$; $p(\nu) = (1-s)p(\mu)$, $\sigma_{\nu} = \sigma_{\mu}$ and $\nu = \mu$ for $\nu = m+1, m+2, \dots, m+n$. Then $p(a|x, \nu) \geq 0$, and σ_{ν} are a collection of subnormalized states that sum up to $s\rho_B + (1-s)\tilde{\rho}_B$ which satisfies $\sum_{\nu} \text{tr}[p(\nu)\sigma_{\nu}] = s \sum_{\lambda} \text{tr}[p(\lambda)\sigma_{\lambda}] + (1-s) \sum_{\mu} \text{tr}[p(\mu)\sigma_{\mu}] = s + (1-s) = 1$. \square

From Lemma 1 we can define the following measure of steerability $\mathcal{S}(\rho)$ and the measure of nonlocality $\mathcal{N}(\rho)$ for a quantum state ρ ,

$$\mathcal{S}(\rho) = \min_{\sigma \in \mathcal{U}} S(\rho || \sigma), \quad (1)$$

and

$$\mathcal{N}(\rho) = \min_{\sigma \in \mathcal{L}} S(\rho || \sigma), \quad (2)$$

where \mathcal{U} and \mathcal{L} stand for the sets of unsteerable states and the LHV states, respectively. $S(\rho || \sigma) = \text{tr}(\rho \log_2 \rho - \rho \log_2 \sigma)$.

The measure $\mathcal{S}(\rho)$ satisfy the following conditions (analogously for $\mathcal{N}(\rho)$): (1) $\mathcal{S}(\rho) \geq 0$, $\mathcal{S}(\rho) = 0$ iff $\rho = \sigma$. (2) Local unitary operations leave \mathcal{S} invariant. (3) For any completely positive trace preserving map Θ , $\mathcal{S}(\Theta\rho) \leq \mathcal{S}(\rho)$ [29–31]. (4) \mathcal{S} is convex, which can be proved in the following way. Let $\rho_1, \rho_2, \sigma_1, \sigma_2$ be four arbitrary states. From the convexity of the quantum relative entropy in both arguments [32], we have

$$\begin{aligned} S(x\rho_1 + (1-x)\rho_2 || x\sigma_1 + (1-x)\sigma_2) \\ \leq xS(\rho_1 || \sigma_1) + (1-x)S(\rho_2 || \sigma_2), \end{aligned}$$

where $x \in [0, 1]$. By definition (1) and Lemma 1, we have

$$\begin{aligned} \mathcal{S}(x\rho_1 + (1-x)\rho_2) \\ \leq S((x\rho_1 + (1-x)\rho_2) || (x\sigma_1^* + (1-x)\sigma_2^*)) \\ \leq xS(\rho_1 || \sigma_1^*) + (1-x)S(\rho_2 || \sigma_2^*) \\ = x\mathcal{S}(\rho_1) + (1-x)\mathcal{S}(\rho_2), \end{aligned}$$

where σ_i^* minimizes $S(\rho_i || \sigma_i^*)$ over $\sigma \in \mathcal{U}$.

Since the steerability is weaker than Bell nonlocality and stronger than nonseparability [7, 26], one has $\mathcal{CC} \subset \mathcal{S} \subset \mathcal{U} \subset \mathcal{L}$. Therefore, we have the following relation:

Theorem 1. *For bipartite quantum states, the following relations hold:*

$$\mathcal{D}(\rho) \geq \mathcal{E}(\rho) \geq \mathcal{S}(\rho) \geq \mathcal{N}(\rho). \quad (3)$$

As an example, let us consider two-qubit Bell-diagonal states. Firstly, for two-qubit pure state $|\phi\rangle = \alpha|00\rangle + \beta|11\rangle$, the quantum discord [33] and quantum entanglement [30, 31] are given by $-\alpha^2 \log_2 \alpha^2 - \beta^2 \log_2 \beta^2$. It has been proven that every entangled pure state is steerable [27], and all entangled pure states violate a single Bell's inequality [34]. Therefore, a separable pure state is unsteerable and LHV. Now consider two-qubit Bell-diagonal states,

$$\rho_{AB} = \frac{1}{4}(I + \sum_{j=1}^3 c_j \sigma_j^A \otimes \sigma_j^B) = \sum_{a,b=0}^1 \lambda_{ab} |\beta_{ab}\rangle \langle \beta_{ab}|, \quad (4)$$

where the σ_j 's are Pauli operators [35]. The eigenstates are the four Bell states $|\beta_{ab}\rangle \equiv (|0, b\rangle + (-1)^a |1, 1 \oplus b\rangle) / \sqrt{2}$, with eigenvalues

$$\lambda_{ab} = \frac{1}{4}[1 + (-1)^a c_1 - (-1)^{a+b} c_2 + (-1)^b c_3].$$

The quantum discord is given by $\mathcal{D}(\rho) = 2 - S(\rho_{AB}) - C$, where $C = \frac{1+c}{2} \log_2(1+c) + \frac{1-c}{2} \log_2(1-c)$ with $c = \max |c_j|$ [33, 35]. For a Bell-diagonal state, when all $\lambda_{ab} \in [0, \frac{1}{2}]$, $\mathcal{E}(\rho_{AB}) = 0$. When $\lambda_{(00)} \geq \frac{1}{2}$ (analogously for other $\lambda_i \geq \frac{1}{2}$), $\mathcal{E}(\rho_{AB}) = \lambda_{(00)} \log_2(\lambda_{(00)}) + (1 - \lambda_{(00)}) \log_2(1 - \lambda_{(00)}) + 1$ [30]. Hence

$$\mathcal{D}(\rho_{AB}) - \mathcal{E}(\rho_{AB}) = 2 - S(\rho_{AB}) - C - E(\rho_{AB}) \geq 0$$

(See Fig. 1).

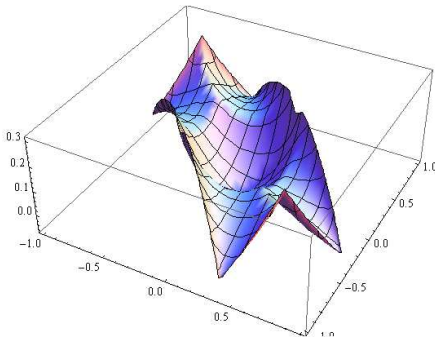


FIG. 1: The quantity $\mathcal{D}(\rho) - \mathcal{E}(\rho)$ for $c_1, c_2 \in [-1, 1]$, $c_3 = 0.4$.

In Ref. [36] it has been proved that any Bell-diagonal state with correlation matrix T is steerable by two projective measurements iff $\lambda_1 + \lambda_2 > 1$, where λ_1 and λ_2

are the two largest eigenvalues of TT^t , where t denotes transpose. For Bell diagonal state (4), the correlation matrix $T = \text{diag}(c_1, c_2, c_3)$. Therefore, $\lambda_1 + \lambda_2$ is the summation of the two largest c_i^2 . Without loss of generality, we assume $c_1 = c_2 = \frac{\sqrt{2}}{2} > |c_3|$. In this case, the Bell-diagonal state is unsteerable, that is, $\mathcal{S}(\rho) = 0$. But since $\lambda_{01} > \frac{1}{2}$, the entanglement is great than zero, $\mathcal{E}(\rho) = \lambda_{01} \log_2 \lambda_{01} - (1 - \lambda_{01}) \log_2(1 - \lambda_{01}) > 0$. For example, for $|c_3| < 0.4$, see Fig. 3.

Ref. [36] showed that a Bell diagonal state is steerable by two projective measurements iff it violates the CHSH inequality. This means that the steerability coincides with NLHV for Bell diagonal states.

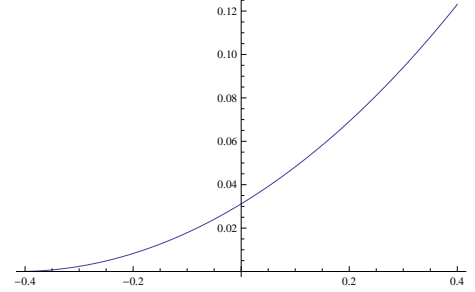


FIG. 2: $\mathcal{E}(\rho)$ for $|c_3| < 0.4$.

CONCLUSIONS AND REMARKS

We have studied some important correlations in bipartite systems. Based on the relative entropy, we presented a unified characterization of discord, entanglement, nonlocality and steerability. For two-qubit states we have showed that the quantities obtained from quantifying nonlocality, steerability, entanglement and discord have strictly monotonic relationship. Detailed investigations have been given to Bell diagonal states. The results can be generalized to the case of multipartite quantum systems. In fact, from geometric point of view, any distance measure can be chosen as a candidate for quantifying quantum correlations. One can choose any possible distance measure instead of $S(\rho || \sigma)$ (relative entropy isn't a distance) in Eq.(1) and (2) to quantify the Bell nonlocality and steerability. For example, we can choose the Bures metric $D(\rho || \sigma) = 2 - 2\sqrt{F(\rho, \sigma)}$, where $F(\rho || \sigma) \equiv [\text{tr}\{\sqrt{\sigma\rho\sigma}\}]^2$ is the so-called fidelity [37].

Moreover, our approach also coincides with other methods in quantifying physical quantities besides quantum correlations, like coherence [18],

$$\mathcal{C}(\rho) = \min_{\sigma \in \mathcal{I}} S(\rho || \sigma) = S(\rho_{\text{diag}}) - S(\rho),$$

where \mathcal{I} and ρ_{diag} stand for incoherent states and the diagonal version of ρ . Instead of (3), one has $\mathcal{C}(\rho) \geq \mathcal{D}(\rho) \geq \mathcal{E}(\rho) \geq \mathcal{S}(\rho) \geq \mathcal{N}(\rho)$. For two-qubit Bell-diagonal

states (4), the diagonal version of ρ_{AB} is $\rho_{(diag)} = \frac{1}{4}[I + c_3\sigma_3 \otimes \sigma_3]$, with eigenvalues $\Lambda_{(1,2)} = \frac{1}{4}[1 + c_3]$ and $\Lambda_{(3,4)} = \frac{1}{4}[1 - c_3]$. Therefore, quantum coherence $\mathcal{C}(\rho) = S(\rho_{(diag)}) - S(\rho_{AB}) = \sum_{a,b} \lambda_{ab} \log_2(\lambda_{ab}) - \sum_{i=1}^4 \Lambda_i \log_2(\Lambda_i)$. Therefore

$$\begin{aligned} \mathcal{C}(\rho_{AB}) - \mathcal{D}(\rho_{AB}) &= \frac{1+c}{2} \log_2(1+c) + \frac{1-c}{2} \log_2(1-c) \\ &\quad - \frac{1+c_3}{2} \log_2\left(\frac{1+c_3}{4}\right) - \frac{1-c_3}{2} \log_2\left(\frac{1-c_3}{4}\right) - 2 \\ &= \frac{1+c}{2} \log_2(1+c) + \frac{1-c}{2} \log_2(1-c) \\ &\quad - \frac{1+c_3}{2} \log_2(1+c_3) - \frac{1-c_3}{2} \log_2(1-c_3) \geq 0. \end{aligned}$$

see Fig. 3.

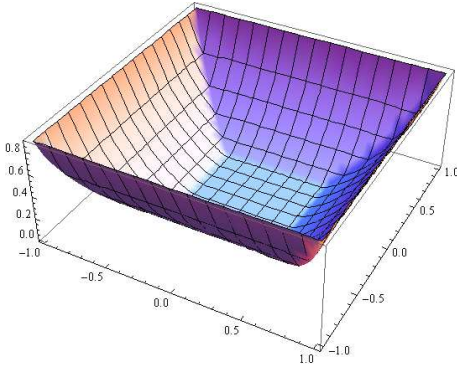


FIG. 3: $\mathcal{C}(\rho_{AB}) - \mathcal{D}(\rho_{AB})$ for $c_1, c_2 \in [-1, 1]$, $c_3 = 0.5$.

Due to the lack of the general expressions for unsteerable states and states admitting LHV models, it is difficult to calculate $\mathcal{S}(\rho)$ and $\mathcal{N}(\rho)$ for a given bipartite state ρ . It is neither an easy task to compute general $\mathcal{E}(\rho)$ (resp. $\mathcal{D}(\rho)$), although in these cases the general expressions of separable states (resp. zero-discord states) are explicitly known. We leave these problems for further investigations.

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[1] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47**, 777 (1935)

[2] E. schrodinger, Ann.Phys. (Berlin) **79**, 361 (1926);**79**, 489 (1926); **80** 437 (1926); **81** 109 (1926)

[3] J.S. Bell, Physics, **1**, 195 (1964)

[4] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009)

[5] R. Chaves, Phys. Rev. Lett. **116**, 010402 (2016)

[6] R. Chaves, C. Majenz and D. Gross, Nat.Comm. **6**, 5766 (2015)

[7] H.M. Wiseman, S.J. Jones and A.C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007)

[8] M.F. Pusey, Phys. Rev. A **88**, 032313 (2013)

[9] H. Zhu, M. Hayashi and L. Chen, Phys. Rev. Lett. **116**, 070403 (2016)

[10] M.T. Quintino, T. Vertesi and N. Brunner, Phys. Rev. Lett. **113**, 160402 (2014)

[11] R. Uola, C. Budroder, O. Gühne and J.P. Pellonpää; Phys. Rev. Lett. **115**, 230402 (2015)

[12] H. Ollivier and W.H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001)

[13] K. Modi, T. Paterek, W. Son, V. Vedral and M. Williamson, Phys. Rev. Lett. **104**, 080501 (2010)

[14] M.B. Plenio and S.F. Huelga, New J. Phys. **10**, 113019 (2008)

[15] F. Levi and F. Mintert, New J. Phys. **16**, 033007 (2014)

[16] M. Horodecki and J. Oppenheim, Nat. Comm. **4**, 2059 (2013)

[17] M. Lostaglio, D. Jennings and T. Rudolph, Nat. Comm. **6**, 6383 (2015)

[18] T. Baumgratz, M. Cramer and M.B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014)

[19] Y. Yao, X. Xiao, L. Gao and C.P. Sun, Phys. Rev. A **92**, 022112 (2015)

[20] O. Gühne, G. Tóth, Physics Reports, **474**, 1 (2009)

[21] L. Henderson and V. Vedral, J. Phys. A **34**, 6899 (2001)

[22] D. Collins et al. Phys. Rev. Lett. **88**, 170405 (2002)

[23] M. Seevinck et al. Phys. Rev. Lett. **89**, 060401 (2002)

[24] J.D. Bancal et al. Phys. Rev. Lett. **103**, 090503 (2009)

[25] S. Luo and S. Fu, Phys. Rev. Lett. **106**, 120401 (2011)

[26] A.C.S. Costa and R.M. Angelo, arxiv: 1510.08030 (2016)

[27] P. Skrzypczyk, M. Navascues and D. Cavalcanti, Phys. Rev. Lett. **112**, 180404 (2014)

[28] I. Kogias, A.R. Lee, S. Ragy, and G. Adesso, Phys. Rev. Lett. **114**, 060403 (2015)

[29] G. Lindblad, Commun. Math. Phys. **40**, 147 (1975)

[30] V. Vedral, M.B. Plenio, M.A. Rippin and P.L. Knight, Phys. Rev. Lett. **78**, 2275 (1997)

[31] V. Vedral, M.B. Plenio, Phys. Rev. A **57**, 1619 (1998)

[32] M.J. Donald, Commun. Math. Phys. **105**, 13 (1986)

[33] S. Luo, Phys. Rev. A **77**, 042303 (2008)

[34] S. Yu, Q. Chen, C. Zhang, C. H. Lai and C. H. Oh Phys. Rev. Lett. **109**, 120402 (2012)

[35] M.D. Lang and C.M. Caves, Phys. Rev. Lett. **105**, 150501 (2010)

[36] Q. Quan et al. Sci.Rep. **6**, 22025 (2016)

[37] R. Josza, J. Mod. Opt. **41**, 2315 (1994)

[38] T. Veřtesi, Phys. Rev. A **78**, 032112 (2008)